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# 2 SEM TDC MTMH (CBCS) C 3

## 2022

( June/July )

## MATHEMATICS

( Core )

Paper : C–3

**( Real Analysis )**

*Full Marks* : 80

*Pass Marks* : 32

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

1\. (a) Define ε-neighbourhood of a point. 1

(b) Find the infimum and supremum, if it exists for the set $A = \{x \in \mathbb{R} : 2x + 5 > 0\}$. 2

*(c)* If

$$S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

then show that $\inf S = 0$, where $\inf S$ denotes the infimum of $S$. 3

*(d)* State and prove that Archimedean Property of real numbers. 4

*(e)* Let $S \subseteq \mathbb{R}$ be a set that is bounded above and for $a \in \mathbb{R}$, $a+S$ is defined as $a+S = \{a+s : s \in S\}$. Show that $\sup(a+S) = a+\sup(S)$, where $\sup(S)$ denotes the supremum of $S$. 5

**2.** *(a)* State the Completeness Property of real numbers. 1

*(b)* Show that

$$\sup\left\{1-\frac{1}{n} : n \in \mathbb{N}\right\} = 1$$ 2

*(c)* Let

$$I_n = \left[0, \frac{1}{n}\right]$$

for $n \in \mathbb{N}$. Prove that

$$\bigcap_{n=1}^{\infty} I_n = 0$$ 3





*(d)* Prove that the set of real numbers is not countable. 4

*Or*

If

$$S = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\right\}$$

find $\inf S$ and $\sup S$.

*(e)* State and prove the nested interval property. 5

*Or*

Prove that there exists a real number $x$ such that $x^2 = 2$.

**3.** *(a)* State the Monotone Subsequence Theorem. 1

*(b)* Show that

$$\lim_{n \to \infty}\left(\frac{n}{n^2+1}\right) = 0$$ 2

*(c)* Show that a convergent sequence of real numbers is bounded. 3

(d) Show that

$$\lim_{n\to\infty} (b^n) = 0$$

if $0 < b < 1$. 4

*Or*

Show that

$$\lim_{n\to\infty} (c^{\frac{1}{n}}) = 1$$

for $c > 1$.

(e) State and prove the Monotone Convergence theorem. 5

*Or*

Let $Y := (y_n)$ be defined as $y_1 = 1$, $y_{n+1} = \frac{1}{4} y_n + 2$, $n \geq 1$. Show that $(y_n)$ is monotone and bounded. Find the limit.

**4.** (a) Give an example of two divergent sequences such that their sum converges. 1

(b) Prove that the limit of a sequence of real numbers is unique. 2





(c) Prove that

$$\lim_{n\to\infty} x_n = 0$$

if and only if

$$\lim_{n\to\infty} (|x_n|) = 0$$ 3

(d) Establish the convergence or divergence of the following sequences (any *one*) : 4

(i) $x_n = \dfrac{(-1)^n n}{n+1}$

(ii) $x_n = \dfrac{n^2}{n+1}$

(iii) $x_n = \dfrac{2n^2+3}{n^2+1}$

(e) Define Cauchy sequence. Prove that a sequence of real numbers is Cauchy if and only if it is convergent. 1+4=5

*Or*

Establish the convergence or divergence of the sequence

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

for $n \in \mathbb{N}$. 5

5. *(a)* State the Cauchy Criterion for convergence of a series. 1

*(b)* Prove that if

$$\sum_{n=1}^{\infty} x_n$$

converges then

$$\lim_{n \to \infty} (x_n) = 0$$ 3

*(c)* Prove that if

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. 3

*(d)* Show that the series

$$\sum_{n=1}^{\infty} x_n$$

converges if and only if the sequence $S = (s_k)$ of partial sums is bounded. 4

*(e)* Define absolute convergence. Show that if a series of real numbers is absolutely convergent then it is convergent. 1+3=4

*(f)* Let $f$ be a positive, decreasing function on $\{t : t \geq 1\}$. Show that the series

$$\sum_{k=1}^{\infty} f(k)$$



*( Continued )*



converges if and only if the improper integral

$$\int_1^{\infty} f(t)dt = \lim_{b \to \infty} \int_1^b f(t)dt$$

exists. 5

*Or*

Show that the series

$$\sum_{n=1}^{\infty} \cos n$$

is divergent.

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