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5 SEM TDC MTMH (CBCS) C 11

2022

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–11

(Multivariate Calculus)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Let partial derivatives of a function of two variables exist. Does it imply that the function is continuous? 1

(b) Find $\frac{\partial f}{\partial x}$, where $f(x, y) = e^{x^2 + xy}$. 1

(c) Show that

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2} & ; \quad (x, y) \neq (0, 0) \\ 0 & ; \quad (x, y) = (0, 0) \end{cases}$$

is continuous at every point, except the origin $(0, 0)$. 3

Or

Using definition, show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & ; \quad (x, y) \neq (0, 0) \\ 0 & ; \quad (x, y) = (0, 0) \end{cases}$$

is continuous at the origin.

(d) Find

$$\frac{\partial^3 u}{\partial z \partial y \partial x} \text{ and } \frac{\partial^3 u}{\partial x^2 \partial y}$$

if $u = \dfrac{x}{y + 2z}$. 3

2. (a) Write True or False : 1

"If a function $f(x, y)$ is continuous at (x_0, y_0), then f is differentiable at (x_0, y_0)."

(b) Use chain rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. 2

(c) Find the values of $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at the point (π, π, π) for the function

$$\sin(x + y) + \sin(y + z) + \sin(x + z) = 0$$ 4

Or

Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\vec{v} = 2\hat{i} - 3\hat{j} + 6\hat{k}$. In what direction does f increases most rapidly at P_0?

3. (a) Find the plane, tangent to the surface $z = x\cos y - ye^x$ at $(0, 0, 0)$. 2

(b) Find the local extreme values of

$$f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$$ 3

(c) Find the points on the hyperbolic cylinder $x^2 - z^2 = 1$ that are closest to the origin. 5

Or

Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

4. (a) Define gradient vector of $f(x, y)$ at a point. 1

(b) Show that

$$\vec{f}(x, y, z) = (y^2 z^3)\hat{i} + (2xyz^3)\hat{j} + (3xy^2 z^2)\hat{k}$$

is a conservative vector field. 2

(c) Calculate the curl $\vec{f}$, where

$$\vec{f} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

2

5. (a) State Fubini's theorem of first form. 1

(b) Evaluate

$$\iint_R f(x, y)\, dx dy \text{ for } f(x, y) = 1 - 6x^2 y^2,$$

$R : 0 \le x \le 1$ and $-2 \le y \le 2$.

2

(c) Prove that

$$\iint_R e^{x^2 + y^2}\, dy dx = \frac{\pi}{2}(e - 1)$$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$.

3

6. (a) Define volume of a region in space. 2

(b) Find $\int_0^2 \int_0^2 \int_0^2 xyz\, dx dy dz$.

2

(c) Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

5

Or

Evaluate the following integral by changing the order of the integration in an appropriate way :

$$\int_0^1\int_0^1\int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz$$

7. (a) Write the formula for triple integral in spherical coordinates. 1

(b) Evaluate : 4

$$\int_0^{\pi}\int_0^1\int_0^{\sqrt{3-r^2}} dz\,r\,dr\,d\theta$$

Or

Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z-1)^2 = 1$.

8. (a) Define Jacobian of a function of two variables. 1

(b) Evaluate : 3

$$\iint\limits_{x^2+y^2\le a^2} (x^2 + y^2)\,dx\,dy$$

(c) Find the value of

$$\int_C \{(x+y^2)\,dx + (x^2 - y)\,dy\}$$

taken in the clockwise sense along the closed curve C formed by $y^3 = x^2$ and the chord joining (0, 0) and (1, 1). 3

Or

Evaluate $\int_C (xy + y + z)\,ds$ along the curve $\vec{r}(t) = 2t\hat{i} + t\hat{j} + (2-2t)\hat{k},\ 0 \le t \le 1$.

9. (a) Define line integrals of a vector field. 1

(b) Find the circulation of the field $\vec{F} = (x-y)\hat{i} + x\hat{j}$ around the circle $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j},\ 0 \le t \le 2\pi$. 3

(c) State and prove the fundamental theorem of line integrals. 4

Or

A fluid's velocity field is $\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$. Find the flow along the helix $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k},\ 0 \le t \le \frac{\pi}{2}$.

10. (a) Define Green's theorem in Tangential form. 1

(b) Evaluate

$$\oint_C (y^2 dx + x^2 dy)$$

using Green's theorem, where C is the triangle bounded by $x=0$, $x+y=1$, $y=0$. 3

(c) State and prove Stoke's theorem. 6

Or

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ by using Stoke's theorem, if $\vec{F} = x^2\hat{i} + 2x\hat{j} + z^2\hat{k}$ and C is the ellipse $4x^2 + y^2 = 4$ in the xy plane, counterclockwise when viewed from above.

(d) Use Divergence theorem to find the outward flux of $\vec{F}$ across the boundary of the region D, where

$$\vec{F} = (y-x)\hat{i} + (z-y)\hat{j} + (y-x)\hat{k}$$

and D is the cube bounded by the planes $x = \pm 1$, $y = \pm 1$ and $z = \pm 1$. 5

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